BMTC-134

ASSIGNMENT BOOKLET

Algebra

School of Sciences
Indira Gandhi National Open University
Maidan Garhi, New Delhi
2025

Dear Student,

This assignment is for the continuous evaluation component of the course and carries a weight of 25%.

Instructions for Formating Your Assignments

Before attempting the assignment please read the following instructions carefully.

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO :..........................

NAME :..........................

ADDRESS :..........................

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COURSE CODE : ....

COURSE TITLE : ....

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.

3) Leave a 4 cm margin on the left, top and bottom of your answer sheet.

4) Your answers should be precise.

5) While solving problems, clearly indicate which part of which question is being solved.

6 **Please solve the assignment by yourself. Please don't copy the answers from the Internet or from your fellow students. If you are found copying the assignment your assignment will be annulled and you will have to submit the assignment for the year 2026.**

7) This assignment has to be submitted before the due date decided by the Study Centre. The assignments received after the due date will not be accepted.

7) This assignment is valid till December 2025. However, if you want to appear in the June/December 2025 Term End Examination submit the assignment before filling up the Term End Examination form. If you fail in the assignment, do the assignment for the year 2026 and submit it.

We strongly suggest that you retain a copy of your answer sheets.

Wish you good luck.

ASSIGNMENT

Course Code: BMTC-134
BMTC-134/TMA/2025
Maximum Marks: 100

1. (a) Define an abelian group. Give an example of a non-abelian group. (You don't need to prove that your example is a group. You have to only prove that it is non-abelian.) (3)

   (b) Define a subgroup of a group. Check whether

$$H = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{C} \right\}$$

   is a subgroup of the the group of 2×3 matrices over $\mathbb{C}$ under addition. (2)

   (c) Define a semigroup. Give an example of an infinite semigroup. (2)

   (d) State Lagrange's theorem. What are the possible orders of subgroups of a group of order 12? (2)

   (e) Let $S = \{1, 2, 3, 4\}$ and $*$ be the binary operation defined by $a * b = a$. Compute the Cayley table for $(S, *)$. Is $*$ commutative? Is $*$ associative? Justify your answers. (6)

2. (a) Let A be a 3×4 real matrix, B be a 4×2 real matrix and C be a 2×3 real matrix. Which of the following operations are defined?

   (i) $CA + B^t$

   (ii) $AB + C^t$

   For those operations that are defined, what is the order of the resulting matrix? (3)

   (b) Let $\alpha = (1\ 2\ 5)$, $\beta = (1\ 4\ 3\ 2) \in S_5$. Compute $\sigma = \alpha \cdot \beta^{-1}$. Write σ as a product of transpositons. What is the signature of σ? (3)

   (c) If F is a field, show that $U(F[x]) = F^*$. (4)

   (d) Let $R = \mathbb{Z}_{20}$.

   (i) Give, with justificaiton, a nilpotent element in R.

   (ii) Give, with justification, a zero divisor in R which is not nilpotent.

   (iii) What is the order of $U(R)$? (5)

3. (a) Let R be a ring in which $a^2 = a$ for all $a \in R$. Show that $a = -a$ and R is commutative. (5)

   (b) Define an integral domain. Give an example of an integral domain which is not a field. (2)

   (c) Calculate the following: (3)

   (i) $\left(\bar{3}x^2 + \bar{4}x + \bar{1}\right) + \left(\bar{3}x^3 + \bar{4}x^2 + \bar{2}x + \bar{3}\right)$ in $\mathbb{Z}_5[x]$.

   (ii) $\left(\bar{3}x^2 + \bar{2}x + \bar{6}\right) \cdot \left(\bar{3}x^3 + \bar{4}x + \bar{5}\right)$ in $\mathbb{Z}_7[x]$.

   (d) Show that, if G is a finite group and $a \in G$, $o(a) \mid o(G)$. Further, show that $a^{o(G)} = e$ for all $a \in G$. Deduce the Euler-Fermat theorem $a^{\phi(n)} \equiv 1 \pmod{n}$ for all $a, n \in \mathbb{N}$, $n \geq 2$, $(a, n) = 1$. (5)

4. (a) Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$. Check whether R is a subring of $M_2(\mathbb{R})$. Is R an ideal of $M_2(\mathbb{R})$? Justify your answer. (4)

(b) Find the gcd of the polynomials $x^4 + 3x + 2$ and $x^3 + 3x^2 + 5x + 3$. (5)

(c) If H and K are normal abelian subgroups of a group, and if $H \cap K = \{e\}$, show that HK is abelian. Will the result be still true if we remove the condition that H and K are normal? Justify your answer. (6)

5. (a) Find the order of each of the elements in $U(15)$. Is $U(15)$ cyclic? Justify your answer. (5)

(b) Let F be a field and let $f(x) \in F[x]$ be irreducible in $F[x]$. Show that the ideal $\langle f(x) \rangle$ is a maximal ideal in $F[x]$. Use this to deduce that $\mathbb{Q}[x]/\langle x^5 + 6x^3 + 12 \rangle$ is a field. (7)

(c) Let $S^1 = \{z \in \mathbb{C}^* \mid |z| = 1\}$ and $U_n = \{z \in \mathbb{C}^* \mid z^n = 1\}$ for $n \in \mathbb{N}$. Check that $U_n \subseteq S^1$. Further, show that $U_n \leq S^1$. (3)

6. (a) Show that $\langle x, 5 \rangle$ is not a principal ideal in $\mathbb{Z}[x]$. (7)

(b) Check whether or not $\langle \bar{3} \rangle$ is a maximal ideal in $\mathbb{Z}_9$. (3)

(c) Let R be a ring (not necessarily commutative) and I and J be ideal of R. Show that $I \cap J$ and $I + J = \{a + b | a \in I, b \in J\}$ are ideals of R. (5)

7. Which of the following statements are true and which are false? Justify your answer with a short proof or a counter example. (10)

(a) Every subgroup of S_3 is normal.

(b) Every abelian group is cyclic.

(c) In a ring with unity, the sum of any two units is a unit.

(d) If a field has characteristic p, p a prime, the field is finite.

(e) If every element in group has finite order, the group is finite.