No. of Printed Pages : 4 **MMT–006**

M. Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) [M. Sc. (MACS)]

Term-End Examination

December, 2024

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 Hours *Maximum Marks : 50*

Weightage : 70%

Note : *(i) Question No.* ***6*** *is compulsory.*

(ii) Attempt any ***four*** *of the remaining questions.*

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1. (a) Is the map $A : \mathbf{R}^3 \to \mathbf{R}^2$, $A(x, y, z) = (x + y, y + z)$ open ? Justify. 2

   (b) Show that the operator A on l^2 defined by $Ax(n) = \frac{i^n}{n} x(n)$ is compact and normal. 2+2

   (c) Prove that c_0 is a closed subspace of c and $\dim \frac{c}{c_0} = 1$. 2+2

2. (a) Show that on a Hilbert space, Hahn-Banach extensions are unique. 2

(b) For $x \in l^1$, define $f(x) = \sum \frac{n}{n+1} x(n)$. Prove that f is a bounded linear functional with $\| f \| = 1$ and there is no $x \neq 0$ with $| f(x) | = \| x \|_1$. 1+1+2

(c) Let M be a closed subspace of a normed space X. Show that $d(x, \mathrm{M}) = \| x \|$ if and only if there is a bounded linear functional $f \neq 0$ on X such that $f = 0$ on M and $| f(x) | = \| f \| \| x \|$. 2+2

3. (a) Define $< \cdot, \cdot > : \mathbf{C}^4 \times \mathbf{C}^4 \to \mathbf{C}$ as :

$$< x, y > = x(1)\overline{y(1)} + x(2)\overline{y(2)} + x(3)\overline{y(3)} - x(4)\overline{y(4)}$$

Show that $< \cdot, \cdot >$ is linear in the 1st variable and conjugate symmetric but not an inner product. 3+3

(b) Define A on C [0, 1] by $\mathrm{A} f(t) = t f(t)$. Prove that A is a bounded operator, but is not compact. 2+2

4. (a) State bounded inverse theorem. Use the theorem to show the following :

Suppose H is a Hilbert space and A is a bounded linear operator such that $\langle Ax, x\rangle \geq 0$ for all $x \in H$. Assume that A defines a complete norm on H given by $\|x\|_A^2 = \langle Ax, x\rangle$.

Show that $\exists$ c such that $\langle Ax, x\rangle \geq c\|x\|^2$.

1+2

(b) Let X be a Banach space and let $c_0(X)$ be the space of all sequences (x_n) in X such that $\lim\|x_n\| = 0$. Prove that $c_0(X)$ is a Banach space with norm $\|(x_n)\| = \sup\|x_n\|$.

3

(c) If M is a closed subspace of a Hilbert space H, determine $M^{\perp\perp}, M^{\perp\perp\perp}$, ... 4

5. (a) Let H be a Hilbert space $u \in H, u \neq 0$. Define $Ax = \langle x, u\rangle u, x \in H$ and

$Bx = \langle x, u\rangle \dfrac{u}{\|u\|}$. Calculate A^*, A^2 and B^2. 3

(b) Let X = C_{00}, $u_n = (0,0,...,0,1,0,0,...,)$, where only the nth entry is 1 and let $k_n = \frac{1}{n}$ for $n = 1,2,.....$ Show that Riesz-Fischer theorem does not hold good. 3

(c) Define $A : l^p \to l^p$ by $Ax = (0, x_1, x_2, ...)$, $1 \le p < \infty$. Show that A is an isometry which has no eigen values. What is R(A) in terms of $\{e_n\}$? 3+1

6. State, with justification, whether the following statements are True *or* False : 5×2=10

(a) If a subspace M of a normed space X is finite dimensional, then $\dim \frac{X'}{M^{\perp}} < \infty$.

(b) A Banach space cannot be the union of countably many proper closed subspaces.

(c) If A is a bounded operator on l^2 and $A^2 = 0$, then A = 0.

(d) There is no linear map $\mathbf{C}^3 \to \mathbf{C}^4$ that takes any orthonormal basis to an orthonormal set.

(e) There is a linear map $\mathbf{R}^m \to \mathbf{R}^n$ with a closed graph that is not continuous.

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