 MMT–008

M. SC. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) [M. SC. (MACS)]

Term-End Examination

December, 2024

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 Hours *Maximum Marks : 100*

Weightage : 50%

Note : *(i) Question No. 8 is compulsory. Attempt any* ***six*** *questions from question nos.* ***1*** *to* ***7****.*

(ii) Use of scientific and non-programmable calculator is allowed.

(iii) Symbols have their usual meanings.

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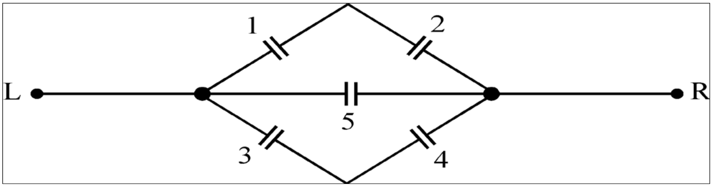
1. (a) Consider the following system consisting of five relays : 1, 2, 3, 4 and 5 between the

terminals L and R. Let the probability of the closing of each relay of the circuit be given by p. If all the relays function independently, what is the probability that current exists between the terminals L and R ? 6



(b) Let the random variables X and Y have the joint probability density function (p.d.f.) :

$$f_{X,Y}(x, y) = \frac{2}{n(n+1)}, \quad y = 1, 2, ..., x;$$
$$x = 1, 2, ..., n$$

Then, compute :

(i) The marginal P.d.f.'s f_X and f_Y.

(ii) The conditional pdf's $f_{X|Y}(\cdot | y)$ and $f_{Y|X}(\cdot | x)$.

(iii) The conditional expectation $E(X | Y = y)$. 9

2. (a) Let the random vector X follows $N_4(\mu, \Sigma)$, where $\mu = (2, 1 - 1 - 2)'$ and

$$\Sigma = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & -1 \\ 1 & -2 & 9 & -1 \\ 1 & -1 & -1 & 16 \end{bmatrix}$$

Find the mean and variance of $P'X$ and $\text{Cov}(P'X, Q'X)$ if $P' = (1\ 1\ 2\ 1)$ and $Q' = (1 - 2\ 2\ 1)$. 8

(b) Define the higher order transition probabilities in a Markov chain. In this context, state and prove the Chapman-Kolmogorov equation which is satisfied by the transition probabilities of a Markov chain. 7

3. (a) In a X-ray clinic, patients arrive at a mean rate of 6 per hour and the technician takes their X-ray at a mean rate of 8 per hour. On the basis of past records, it is assumed

that the service time follows some unspecified positive skewed unimodal distribution with standard deviation of 0.083 hour (approximately 5 minutes). If there is only one technician to serve the patients and the queue capacity is infinite, find the values of the following queuing characteristics : 6

(i) Average queue length

(ii) Average waiting time in the queue

(iii) Average waiting time in the system

(b) In a clinical study of concentration change in fatty acid (FA) and blood glucose (G) in 12 schizophrenic patients and 13 normal persons, the following records were obtained :

	Mean Change	
	Schizophrenics	Normals
G (in mg %)	– 25.6	– 31.1
FA (in m-equiv/litre)	– 0.06	– 0.15

Pooled Covariance matrix

$$S = \begin{bmatrix} 302 & 0.292 \\ 0.292 & 0.0085 \end{bmatrix}$$

Assuming that both FA and G follow a bivariate normal distribution, test the hypothesis that the normal and schizophrenic mean vectors are equal. [You many like to use the following values : 9

$F_{0.05,\,2,\,22} = 3.44$, $F_{0.05,\,2,\,12} = 3.88$]

4. (a) Let the random variables X and Y represent the temperature of a certain object in degrees Celsius and Fahrenheit respectively. Then it is known that $Y = \frac{9}{5}X + 32$ and $X = \frac{5}{9}Y - \frac{160}{9}$.

(i) If $Y \sim N(\mu, \sigma^2)$, determine the distribution of X.

(ii) If $P[90 \le Y \le 95] = 0.95$, then also $P[a \le X \le b] = 0.95$ for some $a < b$.

Determine the numbers a and b. 8

(b) Let $X \sim N_3(\mu, \Sigma)$ with $\mu = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$ and

$\Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Then determine which of the following random variables are independent : 7

(i) (X_1, X_2) and X_3

(ii) $(X_1 + X_2)/3$ and X_3

5. (a) An insurance company has three claims adjusters in its branch office. People with claims against the company are found to arrive in a Poisson fashion at an average rate of 20 per 8-hour day. The amount of time an adjuster spends with a claimant is found to have a negative exponential distribution, with mean service time 40 minutes. Claimants are processed in the order of their appearance : 7

(i) How many hours in a week can an adjuster expect to spend with claimants ?

(ii) How much time, on an average, does a claimant spend in the branch office ?

(b) What are the Renewal Processes ? Give some examples of renewal processes. Mentioning the defining properties of the Poisson process, obtain the probability of n events in a time interval of length t. Also show that inter-arrival times between two successive arrivals in Poisson process follows an exponential distribution. 8

6. (a) Let us consider the vector $Y = (y_1, y_2, y_3, y_4)$ following $N_4(\mu, \Sigma)$ with

$$\mu' = (2\ 6\ 3 - 3) \text{ and } \Sigma = \begin{bmatrix} 16 & 0 & 3 & 0 \\ 0 & 9 & 0 & 2 \\ 3 & 0 & 4 & 1 \\ 0 & 2 & 1 & 9 \end{bmatrix}.$$ 7

(i) Obtain the marginal distribution of $\begin{pmatrix} y_1 \\ y_3 \end{pmatrix}$.

(ii) Obtain the conditional distribution of $\begin{pmatrix} y_3 \\ y_4 \end{pmatrix}$ given $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -2.6 \end{pmatrix}$.

(iii) Find the correlation coefficient between y_1 and y_3.

(b) The transition probability matrix of a two-state Markov chain is given by :

$$P = \begin{bmatrix} (1-\theta) & \theta \\ \beta & (1-\beta) \end{bmatrix}, \quad 0 \le \theta, \beta \le 1$$

$$|1-\theta-\beta| < 1$$

Show that the n-step transition probabilities are given by : 8

$$P^{(n)} = A + (1-\theta-\beta)^n B,$$

where :

$$A = \begin{bmatrix} \beta/(\theta+\beta) & \theta/(\theta+\beta) \\ \beta/(\theta+\beta) & \theta/(\theta+\beta) \end{bmatrix}$$

and $B = \begin{bmatrix} \theta/(\theta+\beta) & -\theta/(\theta+\beta) \\ \beta/(\theta+\beta) & -\beta/(\theta+\beta) \end{bmatrix}.$

7. (a) The life of a certain part in a new automobile is a random variable X whose p.d.f. is negative exponential with parameter $\lambda = 0.005$ days. Then : 8

(i) Find the expected life of the part.

(ii) If the automobile comes with a spare part whose life is a random variable Y, distributed as X and independent of it, find the p.d.f. of the combined life of the part and its spare.

(iii) Find the probability that $X + Y \geq 500$ days.

(b) Consider the function *f* defined by :

$$f(x,y) = \begin{cases} \dfrac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}, & \text{for } (x,y) \text{ outside the square } [-1,1]\times[-1,1] \\ \dfrac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} + \dfrac{1}{2\pi e} x^3 y^3, & \text{for } (x,y) \text{ in the square } [-1,1]\times[-1,1] \end{cases}$$

(i) Show that *f* is a non-bivariate normal p.d.f.

(ii) Show that both marginals are N (0, 1) p.d.f.'s. 7

8. State whether the following statements are True *or* False. Justify your answer with a short proof or a counter example : 10

   (i) Consider an experiment of tossing of a coin of 10 rupees and 5 rupees. Let us define the events :

   E_1 : Head on 10 rupees coin

   E_2 : Head on 5 rupees coin

   E_3 : Coins match

   The events are mutually independent.

   (ii) Let random variable $x \sim$ negative exponential distribution with parameter λ, that is :

   $$f(x) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0$$

   then $P(X > s + t \mid X > s) = P(X > t)$ for some $s, \ t > 0$.

(iii) For the three states 0, 1, 2 the Markov chain has the t. p. m. :

$$\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

with initial distribution $P[X_0 = i] = \frac{1}{3}$, $i = 0, 1, 2$. Then $P[X_2 = 1, X_0 = 0] = \frac{5}{38}$.

(iv) In the Poisson renewal process, N_t, the probability of number of events in $(0, t]$ is given by :

$$P[N_t = r] = F_r(t) - F_{r+1}(t);$$

$F_r(t) = 1 - P[S_r > t]$, S_r being the time of rth event = $X_1 + X_2 + \ldots\ldots + X_r$.

(v) In $(M/M/1) : (\infty / FIFO)$ queueing model, the average queue length is $\frac{\lambda^2}{\mu^2}(\mu - \lambda)^2$.

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