No. of Printed Pages : 5 **RMT–101**

Ph. D. IN MATHEMATICS (PHDMATH)

Term-End Examination

December, 2024

RMT–101 : ALGEBRA

Time : 3 Hours *Maximum Marks : 100*

Note : *(i) There are eight questions in this paper.*

*(ii) The **eighth** question is **compulsory**.*

*(iii) Do any **six** questions from question no. **1** to question no. **7**.*

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1. (a) Let G be an additive abelian group with subgroups H and K. Show that $G \simeq H \oplus K$ if and only if there are homomorphism :

$$H \underset{l_1}{\overset{\pi_1}{\rightleftharpoons}} G \underset{l_2}{\overset{\pi_2}{\rightleftharpoons}} K$$

such that $\pi_1 o\, l_1 = 1_H$, $\pi_2 o\, l_2 = 1_K$, $\pi_1 o\, l_2 = 0$ and $\pi_2 o\, l_1 = 0$, where 0 is the map sending every element onto the zero element, and $l_1 o\, \pi_1(x) + l_2 o\, \pi_2(x) = x$. 8

(b) Let R be a ring with identity. 7

(i) The centre of the matrix that ring $\mathrm{Mat}_n \mathrm{R}$ consists of the matrices $r\mathrm{I}_n$, where r is in the centre of the ring R.

(ii) The centre of $\mathrm{Mat}_n \mathrm{R}$ is isomorphic to the centre of R.

2. (a) Show that a finite abelian group which is not cyclic contains subgroup which is isomorphic to $\mathbf{Z}_p \oplus \mathbf{Z}_p$, where p is a prime. 7

(b) If R is a ring with identity, show that $a \in \mathrm{R}$ is left quasi-regular if and only if $1_{\mathrm{R}} + a$ is left invertible. 4

(c) If R is a ring, not necessarily commutative, and if I is an ideal of R, show that :

$$[\mathrm{R} : \mathrm{I}] = \{r \in \mathrm{R} \mid xr \in \mathrm{I} \text{ for every } x \in \mathrm{R}\}$$

is an ideal of R that contains I. 4

3. (a) Prove that a module A satisfies ascending chain condition if and only if every submodule of A is finitely generated. 5

(b) Show the following conditions on a ring R are equivalent : 6

(i) Every R-module is projective.

(ii) Every short exact sequence of R-module is split exact.

(iii) Every R-module is injective.

(c) Show that any finite group is isomorphic to a subgroup of A_n for some n. 4

4. (a) (i) Let A be a module over a commutative ring R. Show that, for $a \in A, \mathbf{O}_a$, defined by $\mathbf{O}_a = \{r \in R \mid ra = 0\}$ is an ideal of R. If $\mathbf{O}_a \neq \{0\}$, then a is called a torsion element in R.

(ii) If R is an integral domain, show that T (A), the set of all torsion elements of A, is a submodule of A.

(iii) Show that (ii) may false if R is not an integral domain. 5

(b) Let G be a non-abelian group of order $p^3 (p$ prime). Show that the centre of the group is the subgroup generated by all elements of the form $aba^{-1}b^{-1}$. 5

(c) Determine the structure of the abelian group G defined by generators a, b and c and relations $6a + 12b = 0$ and $3a + 9b - 3c = 0$. 5

5. (a) Let R be the ring of all rational numbers with odd denominators. Show that the Jacobson radical J (R) of R is an ideal consisting of all elements in R with even numerators and odd denominators. 5

(b) Find the elementary divisors and invariant factors of the group : 6

$$G = \mathbf{Z}_{26} \oplus \mathbf{Z}_{42} \oplus \mathbf{Z}_{49} \oplus \mathbf{Z}_{200} \oplus \mathbf{Z}_{1000}$$

(c) A subgroup H of a group G is called a characteristic subgroup of G if every automorphism of G carries H to itself. 4

(i) Show that every characteristic subgroup of a group is normal.

(ii) Show that the centre of any group is a characteristic subgroup.

6. (a) Prove the following :

(i) If A is a torsion abelian group, $\mathbf{Q} \otimes A = 0$. 2

(ii) $\mathbf{Z}_m \otimes \mathbf{Z}_n \simeq \mathbf{Z}_d$, where d is the g.c.d. of m and n. 7

(b) Find all the Sylow 2-subgroups of A_4. 6

7. (a) If F is a field, then every non-zero element of F[[x]] is of the form $x^k u$, where u is a unit and k is an integer. 5

(b) Show that every group of prime power order is nilpotent and every finite abelian group is nilpotent. 6

(c) Find the normaliser of the element $\begin{bmatrix} \bar{2} & 0 \\ 0 & \bar{3} \end{bmatrix}$ in $GL_2(\mathbf{Z}_5)$. 4

8. Which of the following are true and which are false ? Give reasons for your answers : 10

(i) Any two Sylow-p subgroups of a finite group are isomorphic.

(ii) The multiplicative group of non-zero rational numbers is a free abelian group.

(iii) S_3 is indecomposable.

(iv) If R is a cumulative ring and M is a maximal ideal, then R / M is a field.

(v) For any ring R, a finitely generated R module is also finitely generated as an abelian group.

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