# Ph. D. PROGRAMME IN MATHEMATICS (PHDMT)

## Term-End Examination

## December, 2024

### RMT-102 : ANALYSIS

*Time : 3 Hours* *Maximum Marks : 100*

***Note :*** *Marks are indicated against each question or part thereof. Question No.* ***1*** *is compulsory. Attempt as many questions as you can from Question Nos.* ***2*** *to* ***8****. The total marks awarded will be 100.*

1. Which of the following statements are true *or* false ? Justify your answer by giving a short proof of the statement which you think is true or by illustrating with counter-examples for the statements which are false : 5×2=10

   (i) The product of two harmonic functions is harmonic.

(ii) If $f(z)$ is analytic on a domain D and Re $f(z)$ is constant, then $f(z)$ is constant.

(iii) A closed and bounded set in a metric space is compact.

(iv) All metric spaces are normed linear spaces.

(v) The function $f : \mathbf{R} \to \mathbf{R}$ defined by :

$$f(x) = \begin{Bmatrix} 1, & \text{if } x \in \mathrm{Q} \\ 0, & \text{if } x \notin \mathrm{Q} \end{Bmatrix}$$

where, Q is the set of rationals, is Lebesgue integrable.

2. (a) Let (X, $d_1$), (Y, $d_2$) be two metric spaces and $f : \mathrm{X} \to \mathrm{Y}$. Show that $f$ is continuous at $x_0$ if and only if for every sequence $\{x_n\}$ in X converging to $x_0$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$ in Y. 5

(b) State Fatou's lemma. Show by an example that the inequality in the lemma cannot be replaced by equality. 3

(c) Let $X = \mathbf{R}^2$ with discrete metric space $d$. Then show that $\left\{\left(\frac{1}{n}, \frac{1}{n}\right)\right\}_{n=1}^{\infty}$ does not converge to (0, 0). 2

(d) Consider the Banach space $C\{0, 1\}$ (with the sup norm). For any fixed $g \in C\{0, 1\}$, define a map $\lambda_g$ on C{0, 1} such that : 5

$$\lambda_g(f) = \int_0^1 f(t)\, g(t)\, dt$$

Show that $\lambda_g$ is a bounded linear functional. Find $\|\lambda_g\|$.

3. (a) Show that $u(x, y) = \sinh x \sin y$ is harmonic in the entire complex plane **C**. Also find the harmonic conjugate function of $u$. Is the harmonic conjugate function unique ? Justify your answer. 5

(b) Prove that every Mobius transformation $T : \mathbf{C}_\infty \to \mathbf{C}_\infty$ has at most two fixed points in $\mathbf{C}_\infty$ under $T \equiv I$. 4

(c) If a function $f = u + iv$ in analytic in a domain $\Omega$ such that $v = u^2$, then prove that $f$ is a constant. 3

(d) Find the maximum modulus of $f(z) = e^y + 1$ on $|z| \leq 1$. 3

4. (a) Find the Mobius transformation which maps 0 to $i$, $i$ to $(-1)$ and $\infty$ to 1. 3

(b) Let $u : G \to \mathbf{R}$ be harmonic function and let $\overline{B}(a, r)$ be a closed disc contained in G. If $\gamma$ is the circle $|z - a| = r$, then show that : 5

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + r e^{i\theta})\, d\theta$$

(c) Let $f \in H(\Omega)$ and $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. Show that $f$ is conformal at $Z_0$. 7

5. (a) If a set E has finite measure, then show that $L^p(E) \subset L'(E)$. 3

(b) Show that a countable union of open sets in a metric space is open. 3

(c) Let X be a non-empty set and let :

$\mathbf{F}_1 = \{A \subset X : A \text{ is countable or } A^C \text{ is countable}\}$

Check whether $\mathbf{F}_1$ is a $\sigma$-algebra or not. 4

(d) Let $X = l^{\infty}$, the set of all bounded scalar sequences. Define a function $\| . \|_{\infty} : X \to \mathbf{C}$ defined by $\| x \| = \sup_{n} | x_n |$. Show that X is a normed linear space under $\| . \|_{\infty}$. Is X an inner product space ? Justify. 5

6. (a) Let X be a normed linear space over K. Let $x_n \in X$ for $n$ = 1, 2, .... , $\lambda_n \in K$. Suppose that $x_n \to x$ in X and $\lambda_n \to \lambda$ in K. Then show that : 5

(i) $\| x_n \| \to \| x \|$ as $n \to \infty$

(ii) $\{x_n\}$ is bounded

(iii) $\lambda x_n \to \lambda x$ as $n \to \infty$

(b) Show that $C_{00}$ as a subspace of $l^{\infty}$ is not a Banach space. 4

(c) State open mapping theorem and use it to prove closed graph theorem. 6

7. (a) Let φ be a simple function. Then show that : 6

(i) $\int_{A \cup B} \varphi \, dm = \int_A \varphi \, dm + \int_B \varphi \, dm$

where A and B are disjoint measurable sets.

(ii) $\int_E a\varphi \, dm = a \int \varphi \, dm$

where $a \in \mathbf{R}$ is a constant.

(b) State dominated convergence theorem. Use the theorem to find : 6

$$\lim_{n \to \infty} \int_1^{\infty} f_n(x) \, dx$$

where $f_n(x) = \dfrac{\sqrt{x}}{1 + nx^3}$.

(c) Give an example of an algebra which is not a σ-algebra. 3

8. (a) When is a real-valued function said to be measureable ? Check whether the following function is Lebesgue measurable : 3

$$f(x) = \begin{cases} x^2, & \text{if } x \in Q \\ -x^2, & \text{if } x \notin Q \end{cases}$$

(b) Let $d$ be a function defined from $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by : 3

$$d\big((x_1, x_2), (y_1, y_2)\big) = |x_1 - y_1| + |x_2 - y_2|$$

Check whether $d$ is a metric.

(c) Let X be a compact metric space and $E \subseteq X$. If E is closed, then show that E is compact. Is the converse of this statement true ? Justify your answer. 6

(d) State Hahn-Banach Extension theorem. Illustrate with an example. 3

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