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RMTE-105

Ph. D. (MATHEMATICS)

(PHDMT)

Term-End Examination

December, 2022

RMTE-105 : PARTIAL DIFFERENTIAL EQUATIONS (ELECTIVE)

Time : 3 Hours *Maximum Marks : 100*

Note : (*i*) *Question No.* ***1*** *is compulsory.*

(*ii*) *Answer any* ***nine*** *questions out of remaining Q. No.* ***2*** *to* ***12****.*

(*iii*) *Use of scientific (non-programmable) calculator is allowed.*

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1. State whether the following statements are True *or* False. Justify your answer with the help of a short proof or a counter-example. No marks will be awarded without justification :

$2 \times 5 = 10$

(a) If L denotes Laplace transform and if

$$L\{f(t)\} = F(s) \quad \text{and} \quad G(t) = \begin{cases} f(t-\alpha), & t > \alpha \\ 0 \quad , & t < \alpha \end{cases}$$

then $L\{G(t)\} = e^{-\alpha s} F(s)$.

(b) The partial differential equation :

$$\frac{\partial^2 u}{\partial x^2} + 4x \frac{\partial^2 u}{\partial y \partial x} + (1 - y^2) \frac{\partial^2 u}{\partial y^2} = 0$$

is parabolic inside the ellipse $4x^2 + y^2 = 1$.

(c) While solving a partial differential equation using a variable separable method, we equate the ratio to a constant which must be a negative integer.

(d) Cauchy characteristic method can be used to solve any first order p.d.e.

(e) Any non-linear p.d.e. can be converted into linear p.d.e. by using Cole-Hopf transformation.

2. Using method of characteristics, determine the solution of the equation : 10

$$u_{x_1} u_{x_2} = u \text{ in } \cup$$

$$u = x_1^3 + 1 \text{ on } \Gamma$$

where $\cup$ is the half-space $\{x_1 > 0\}$ and $\Gamma = \{x_1 = 0\}$.

3. Explain for what type of equation, we can apply Cole-Hopf transformation. Using Cole-Hopf transformation, determine the solution of the following equation : 10

$$u_t - 3u_{xx} + \frac{1}{5}u_x^2 = 0 \text{ in } \mathbf{R} \times (0, \infty)$$

$$u = x^2 + 1 \text{ on } \mathbf{R} \times \{t = 0\}$$

4. Solve the following equation by using Laplace transform : 10

$$\frac{\partial u}{\partial x} = 2\left(\frac{\partial u}{\partial t}\right) + u, \; x > 0, \; t > 0$$

$$u(x, 0) = 6e^{-3x}.$$

5. Explain and define second order parabolic equation in U × (0, T), where T > 0 and $U \subseteq \mathbf{R}^n$. Give an example of second order parabolic equation in $\mathbf{R}^2 \times (0, T)$. 10

6. Define integral surface for a semi-linear Cauchy problem. Using Cauchy characteristic method, determine the solution of the equation : 10

$$xu_x + yu_y = xe^{-u}, \; u(x, x^2) = x.$$

7. Determine an explicit formula for a function u solving the initial value problem : 10

$$u_t + b \cdot \mathrm{D}u = f \text{ in } \mathbf{R}^n \times (0, \infty)$$

$$u = x^2 \text{ on } \mathbf{R}^n \times \{t = 0\}$$

where $b \in \mathbf{R}^n$ and $f_i\, \mathbf{R}^n \times (0, \infty) \to \mathbf{R}$ are given.

8. Prove that there exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that : 10

$$| a(u,v) | \leq \alpha || u ||_{\mathrm{H}'_0(\mathrm{U})} || v ||_{\mathrm{H}'_0(\mathrm{U})}$$

$$\text{and } \beta || u ||^2_{\mathrm{H}'_0(\mathrm{U})} \leq a(u,u) + \nu || u ||^2_{\mathrm{L}^2(\mathrm{U})}$$

for all $u, v \in \mathrm{H}'_0(\mathrm{U})$.

9. Explain weak solution of second order elliptic equation in $\mathbf{R}^n$. 10

10. State Lax-Milgram theorem and explain how one can apply it to prove the existence and uniqueness of weak solution of second order elliptic equation. 10

11. Define envelopes. State and prove the theorem of construction of new solution by using envelope. 10

12. Explain coercivity of a bilinear form. Show that : 6+4

$$a\big((x_1, x_2), (y_1, y_2)\big) := 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$$

is a coercive bilinear form.

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