No. of Printed Pages : 8 **MMT–004**

M. SC. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

[M. SC. (MACS)]

Term-End Examination

December, 2025

MMT-004 : REAL ANALYSIS

Time : 2 Hours *Maximum Marks : 50*

Note : *(i) Question No.* ***1*** *is compulsory.*

(ii) Attempt any ***four*** *questions out of Question Nos.* ***2*** *to* ***6****.*

(iii) Use of calculator is not allowed.

(iv) Notations as in the study material.

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1. State whether the following statements are True *or* False. Justify your answers :

5×2=10

(a) If T = $[a, b]$ and B(T) denote the set of all bounded real valued functions on T, for f, g, $\in$ B(T) define :

$$d(f, g) = | f(t_0) - g(t_0) |$$

for some fixed $t_0 \in$ T. Then d is a metric on B(T).

(b) The sequence $\left\{\frac{1}{n}\right\}$ is a convergent Cauchy sequence in (0, 1) under the discrete metric on (0, 1).

(c) If a subset of a metric space is bounded, then it is totally bounded.

(d) The function $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(x, y) = (e^x, e^y, x + 4)$ is continuously differentiable.

(e) The sequence $f_n : \chi_{[n, n+1]}, n = 1, 2, 3, ...$ satisfy all the conditions of the monotone convergence theorem.

2\. (a) Suppose A and B are non-empty disjoint closed subsets of a metric space (X, d). Prove that there exist open sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$. 3

(b) Consider C[0, 1] with the metric defined by the Riemann integral. Show that the sequence given by :

$$f_n(x) = \begin{cases} 1 & , \text{ if } 0 \le x \le \frac{1}{2} \\ -2^n\left(x - \frac{1}{2}\right) + 1, & \text{ if } \frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{2^n} \\ 0 & , \text{ if } \frac{1}{2} + \frac{1}{2^n} \le x \le 1 \end{cases}$$

is a Cauchy sequence which is not convergent. 4

(c) Define the following with an example of each : 3

(i) A compact set in a metric space

(ii) A connected set in a metric space

3. (a) Let $f : X \to \mathbb{R}$ be a continuous function when X is a metric space. Suppose X is connected. Prove that $f(X)$ is an interval. 2

(b) Define the differentiability of a map $f : E \to \mathbb{R}^m$, E is an open set in $\mathbb{R}^n$, at a point $x \in E$. Prove that if the derivative exists, then it is unique.

Check whether the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by :

$$f(x, y) = \left(\frac{1}{x}, \ y\right), (x, y) \neq (0, 0)$$

$$(0, \ 0) \ , (x, y) = (0, 0)$$

is differentiable or not at (0, 0). 4

(c) Define an algebra and a σ-algebra on a non-empty set X. Prove through an example, that an algebra may fail to be a σ-algebra. 4

4. (a) Compute the derivative of the function $f : \mathbb{R}^4 \to \mathbb{R}^4$ given by :

$$f(x_1, x_2, x_3, x_4)$$

$$= (x_1^2 - x_2^2, 2\, x_1 x_2, x_1 x_3, x_1^2 x_3^2 x_4^2)$$

at (1, 2, – 1, – 2). Find also the directional derivative $D_v f$ (1, 2, – 1, – 2) with v = (2, 1, – 2, – 1). 4

(b) Check whether the function $f : \mathbb{R}^4 \to \mathbb{R}^4$ defined by :

$$f(x, y, z, w) = (3x + 2y, x^2 - y^2, wz, y + w)$$

is locally invertible at the point (1, 2, 1, 1). 3

(c) Define Lebesgue outer measure and show that the Lebesgue outer measure of a countable set is 0. 3

5. (a) State the Implicit Function Theorem for a function $f : \mathrm{E} \subset \mathbb{R}^{n+m} \to \mathbb{R}^n$, where E is open. 2

(b) Show that : 2

$$\overline{\mathrm{A}} = \{x \in \mathrm{X} : \mathrm{B}(x, r) \cap \mathrm{A} \neq \phi \text{ for every } n > 0\}$$

(c) Use the dominated convergence theorem to find :

$$\lim_{n\to\infty}\int_1^{\infty} f_n(x)\,dx$$

where :

$$f_n(x) = \frac{\sqrt{x}}{1 + nx^5}$$ 4

(d) Is a continuous function measurable ? What about the converse ? Justify. (The function is real valued with a measurable domain). 2

6. (a) Prove that the continuous image of a path connected metric space is connected. 3

(b) Find the Fourier series for $f(t) = t$ on $[-\pi, \pi]$. 3

(c) Find the critical points of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by : 4

$$f(x, y, z) = x^2y^2 + z^2 + 2x - 4y + z$$

and classify them.

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