No. of Printed Pages : 6 **MMT–003**

M. SC. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

[M. SC. (MACS)]

Term-End Examination

June, 2025

MMT-003 : ALGEBRA

Time : 2 Hours *Maximum Marks : 50*

Note : *(i) Question No.* ***1*** *is compulsory.*

(ii) Answer any ***four*** *questions from Q. Nos.* ***2*** *to* ***6****.*

(iii) Calculator is not allowed.

1. State whether the following statements are true *or* false, giving reasons for your answer :

(a) If G is a group of order $p^5 q$, p, q distinct primes and if $q \not\equiv 1 \pmod{p}$, then G has a unique sylow p-subgroup. 2

(b) The field $Q\left(\sqrt[3]{2}, \sqrt{7}, \sqrt[4]{3}\right)$ is a radical extension of Q. 2

(c) There are 3 units in $\mathbf{Z} + \mathbf{Z}_w \subseteq \mathbf{C}$, where $w = e^{\frac{2\pi i}{3}}$. 2

(d) If a group of order 21 acts on a set with 8 elements, then it must fix at least one element of the set. 2

(e) If $a, b \in \mathbf{Z}$ and ab is a square mod p, then a and b are squares mod p. 2

2. (a) Give an example of an element of order 35 in the permutation group S_{13}. 2

(b) Let K be a Galois extension of a field F with Galois group the quarternion group of order 8. How many fields L are there such that $F \underset{\neq}{\subset} L \underset{\neq}{\subset} K$ and how many of these L will be normal over F ? Further, give the degrees of each of the subfields. Justify your answers. 5

(c) What is the units digit (in the decimal system) of 13^{1025} ? 3

3. (a) Solve the following simultaneous system of congruences : 4

$$x \equiv 2 \pmod{5}$$

$$x \equiv 1 \pmod{7}$$

$$x \equiv 3 \pmod{11}.$$

(b) Is $\sqrt[4]{5}+\sqrt[4]{7}$ a constructible number ? Justify your answer. 3

(c) Write the class equation of a finite group G. Is there a group of order 10 with class equation 10 = 1 + 2 + 2 + 5. If yes, exhibit the group and its conjugacy classes. If no, give the class equation of S_3. 3

4. (a) Define the symplectic group $SP_{2n}(\mathbf{R})$ and show that $SP_2(\mathbf{R}) = SL_2(\mathbf{R})$. Is it true that in general $SP_{2n}(\mathbf{R})^2 = SL_{2n}(\mathbf{R})$? 3

(b) Find an element α such that $Q(\alpha) = Q\left(\sqrt{3}, \sqrt{7}\right)$. Justify your choice of α. 3

(c) If G is a finite group and P is a Sylow p-subgroup of G, prove that : 4

$$N_G(N_G(P)) = N_G(P)$$

5\. (a) Define the characteristic of a field. Show that the characteristic of a field is zero or a prime number. 3

(b) Determine all possible abelian groups of order 500 (upto isomorphism) giving the invariant factors of each group. 4

(c) Define nilpotent element in a ring. Check whether $\overline{3}$ and $\overline{15}$ are nilpotent in $\mathbf{Z}_{45}$. 3

6\. (a) What is the degree of the field Q[α] over Q, where α is a root of $x^8 - 2$? Justify your answer. 2

(b) Let G be a group. Show that Inn (G), the group of inner automorphisms of G, is a normal subgroup Aut (G), the group of all automorphisms of G. 3

(c) Show that $< 5, x >$ is not a principal ideal in $\mathbf{Z}[x]$. 5

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