No. of Printed Pages : 5 **MMT–006**

M. SC. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) [M. SC. (MACS)]

Term-End Examination

June, 2025

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 Hours *Maximum Marks : 50*

Weightage : 70%

Note : *(i) Question No.* ***1*** *is compulsory.*

(ii) Answer any ***four*** *questions from the remaining questions.*

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1. State, with justification, whether each of the following statements is true *or* false : 5×2=10

(a) $[-2, 2] \times [-2, 2]$ is the unit ball for some norm on $\mathbf{R}^2$.

(b) If M is a proper subspace of a Hilbert space H, then $M^{\perp} \neq (0)$.

(c) Every non-zero linear functional on a normed space is an open map.

(d) The dual of a separable normed space is separable.

(e) For a non-zero compact operator, 0 is never an eigen value.

2. (a) State the Schwarz inequality and find conditions for equality to hold. 1+3=4

(b) Prove the completeness of the space $\left(l^1, \|\cdot\|_1\right)$. 3

(c) If $\{A_n\}$ is a sequence of bounded linear operators on a Banach space X and if $A_n x \to Ax \; \forall \, x \in X$, show that A is linear and continuous. 3

3. (a) Show that the canonical map $J : X \to X''$ for a normed space X is a

linear isometry. Give an example, where it is not onto. 2+2=4

(b) A is a bounded linear operator on a Hilbert space H. If A is unitary, then for every orthonormal basis $\{u_i\}$ of H, $\{Au_i\}$ and $\{A^*u_i\}$ are both orthnormal bases for H. 3

(c) For $f \in C[0,1]$ define $\varphi(f) = \int_0^1 x\, f(x)\, dx$, find $\|\varphi\|$ for the norm $\|.\|_\infty$ on C [0, 1]. 3

4. (a) If a Banach space is reflexive, then prove that its dual is reflexive. 3

(b) Let A be a bounded self-adjoint operator on a Hilbert space H. Prove that : 3

$$\|A\| = \sup\{|< Ax, x >| : \|x\| = 1\}$$

(c) Show that a linear map T on a Banach space X is closed if and only if the graph G (T) is complete in X × X. 2+2=4

5. (a) State the Riesz representation theorem for Hilbert spaces. If f is a bounded linear functional on a Hilbert space H, then show that the representing vector of f is given by $\Sigma\overline{f(e_i)}e_i$, where $\{e_i\}$ is an orthonormal basis. 1+3=4

(b) If $\{x_n\}$ is a sequence in a normed space X such that $\{f(x_n)\}$ is convergent for each $f \in X'$, prove that $\{x_n\}$ is bounded. 3

(c) Let H be a Hilbert space and let $u, v \in H$. Define $Ax = \langle x, u \rangle v$. Calculate $\|A\|$. 3

6. (a) For $A, B \in M(n, \mathbf{R})$, the space of $n \times n$ real matrices, define $\langle A, B \rangle = \text{trace}(AB^t)$. Show that this gives an inner product and find an orthonormal basis. (B^t = transpose of B). 2+2=4

(b) Let $\{x_1,.......,x_n\}$ be a linearly independent set in a normed space X. Prove that there are $f_i \in X'$ such that $f_i(x_j) = \delta_{ij}$. 3

(c) Prove that $l^1 \subset l^2$ and find $(l^1)^\perp$ in l^2. 2+1=3

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